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Some Characterization of Harmonic Convex and Harmonic Quasi-convex functions in Complex Plane

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Abstract - The approach in this paper is essentially based on the results which have already proved in the case of real numbers. We have introduced a new class of convex functions in the complex plane and thereby defined the harmonically complex convex function.

In the paper, we have proved that a harmonically complex log-convex functions implies harmonically complex convex functions. It is, further proved that harmonically complex convex functions implies the harmonically complex quasi-convex functions. Beside this, we have also defined some properties of harmonically complex convex functions and harmonically complex quasi-convex functions. The ideas and techniques used in this paper are very interesting and may open new dimensions to the researcher for further research in this field.

Keywords - Complex convex functions, harmonically complex convex functions, harmonically complex quasi-convex functions, harmonically complex log-convex functions, harmonically complex pseudo-convex functions.

I. Introduction

Theory of Convexity is a glowing topic of research in the recent years; much attention has been given to it by researchers. It plays an important role in the development of various fields of pure and applied mathematics. Several new generalizations and extensions of classical convexity theory have been proposed and investigated, which opens new door for researchers in this area. Due to its symmetry in shape, Convexity theory plays a pivotal role in various fields of mathematics like differentiable geometry, topology, complex analysis, optimization theory and non-linear programming. A significant class of convex functions, called harmonic convex was introduced by Anderson et al. [1] and Iscan [4]. Noor and Noor [6, 7] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequalities, which is called the harmonic variational inequality. For recent developments and applications, see [5-7, 8-13, 19-20].

To the best of my knowledge, this field is new one and has not been fully developed yet. In this paper, we have shown that harmonic convex and harmonic quasi convex functions in complex plane have some nice properties. We have investigated various fundamental properties of harmonic convex functions in complex plane by taking up the same ideas and techniques used in one dimension.

II. Preliminaries

In this section, we recall some basic results and define the concept of harmonically convex and harmonically quasi-convex functions in complex plane.

Definition 2.1. A set $S \subseteq \mathbb{C}$ is said to be convex set, if for every p, q $\in S$ such that $p = a+ib$, $q = c+id$, where $a,b,c,d \in R$ we have $p+t(q-p) \in S,\ \forall t \in [0,1]$.

Definition 2.2. A function $f: S \subseteq \mathbb{C} \to \mathbb{C}$ is said to be complex convex function, if $f(p+t(q-p)) \leq (1-t)f(p)+tf(q),\ \forall\, p,q \in S \; and \; t \in [0,1]$.

Definition 2.3. [8] A set $S \subseteq \mathbb{C}_+$ *such that* $\mathbb{C}_+ = \{a+ib : a>0, b>0\}$ is said to be complex harmonic convex set, if

$$\frac{pq}{q+t(p-q)} \in S \;\forall\, p,q \in S, t \in [0,1].$$

Definition 2.4. [8] A function $f: S \subseteq \mathbb{C}_+ \to \mathbb{C}$ is said to be complex harmonic convex function if

$$f\left(\frac{pq}{q+t(p-q)}\right) \leq (1-t)f(p)+tf(q) \;\forall\, p,q \in S, t \in [0,1].$$

Definition 2.5. The function f is said to be complex harmonic concave function if and only if $-f$ is complex harmonic convexfunction.

Definition 2.6. A function $f: S \subseteq \mathbb{C}_+ \to \mathbb{C}$ is said to be complex harmonic quasi-convex function if

$$f\left(\frac{pq}{q+t(p-q)}\right) \leq \left\{ f(p), f(q) \right\} \forall\, p,q \in S, t \in [0,1].$$

Definition 2.7. (a) The function f is said to be complex harmonic quasi-concave if and only if $-f$ is complex harmonic quasi-convex.

(a) A function f is complex harmonic quasi-convex, if whenever $f(q) \geq f(p)$

(b) A function f is said to be strictly complex harmonic quasi-convex, if $f(q) > f(p)$

Definition 2.8. [3] A function $f: S \subseteq \mathbb{C}_+ \to \mathbb{C}$ is said to be complex harmonic log-convex function if

$$f\left(\frac{pq}{q+t(p-q)}\right) \leq \left(f(p)^{1-t} . f(q)^{t} \right) \forall\, p,q \in S, t \in [0,1].$$

Definition 2.9. [2] Let S be a non-empty in $\mathbb{C}^n$ and $f : S \subseteq \mathbb{C} \to \mathbb{C}$ be a function. Then epigraph of f denoted by $E(f)$ and is defined by $E(f) = \{(p,\lambda) : p \in S, \lambda \in \square, f(p) \le \lambda\}$

Definition 2.10. A function $f : S \subseteq \mathbb{C}_+ \to \mathbb{C}$ is said to be complex harmonic pseudo-convex function with respect to a strictly positive function $\eta(.,.)$ such that $f(p) > f(q)$

$$\Rightarrow f\left(\frac{pq}{q+t(p-q)}\right) < f(p) + t(1-t)\eta(p,q) \, \forall\, p,q \in S, t \in (0,1).$$

Definition 2.11. Let S be a non-empty in $\mathbb{C}_+$. Then the function $f : S \to \mathbb{C}$ is said to be

(a) Complex harmonic pseudo-convex function, if $\forall\, p,q \in S$ with $\left\langle f'(q), \frac{pq}{q-p} \right\rangle \ge 0$, we have $f(p) \ge f(q)$

(b) complex harmonic pseudo quasi-convex function, if $\forall\, p,q \in S$ with $f(p) \le f(q)$, we have $\left\langle f'(q), \frac{pq}{q-p} \right\rangle \le 0$

Theorem 2.12. *Let S be a complex harmonic convex set and $f : S \subseteq \mathbb{C}_+ \to \mathbb{C}$ be a complex harmonic convex function. Then any local minimum of f is a global minimum.*

Proof. Let $p \in S$ be a local minimum of a complex harmonic convex function f.
Suppose on the contrary that $f(q) < f(p), q \in S$, since f is complex harmonic convex function. Then,

$$\begin{aligned} f\left(\frac{pq}{q+t(p-q}\right) &\le (1-t)f(p) + tf(q), \forall\, p,q \in S, t \in (0,1) \\ &\le f(p) - tf(p) + tf(q) \\ &= f(p) + t(f(q) - f(p)) \\ &\le t[f(q) - f(p)] \end{aligned}$$

$$\therefore f\left(\frac{pq}{q+t(p-q}\right) - f(p) \le t[f(q) - f(p)]$$

For some t>0, it follows that $f\left(\frac{pq}{q+t(p-q}\right) < f(p)$, which is a contradiction.

Hence every local minimum of f is global minimum.

Theorem 2.13. *If $f: S \subseteq \mathbb{C}_+ \to \mathbb{C}$ be complex harmonic log convex function on S, then f is complex harmonic convex function implies f is complex harmonic quasi convex function.*

Proof. Suppose f is complex harmonic log convex function. Then for all $p,q \in S$ and $t \in [0,1]$, we have

$$f\left(\frac{pq}{q+t(p-q}\right) \leq (f(p))^{1-t}.(f(q))^{t}$$

$$\leq (f(p))^{1-t} + (f(q))^{t}$$

$$\leq (1-t)f(p) + tf(q)$$

$$\leq \max\{f(p), f(q)\}.$$

This proves that f is complex harmonic log convex function

$\Rightarrow$ f is complex harmonic convex function

$\Rightarrow$ f is complex harmonic quasi convex function.

The converse of the theorem (2.13) need not be true.

III. Main Result

In this section, we discuss some properties of complex harmonic convex function and complex harmonic quasi convex function.

Theorem 3.1. *If S_1 and S_2 are two complex harmonic convex sets, then $S_1 \cap S_2$ is also a complex harmonic convex set.*

Proof. Let $p,q \in S_1 \cap S_2, t \in [0,1]$. We have to prove that $S_1 \cap S_2$ *is also a complex harmonic convex set*. Then $p,q \in S_1 \cap S_2$

$\Rightarrow p,q \in S_1$ *and* $p,q \in S_2$

$$\Rightarrow \frac{pq}{q+t(p-q)} \in S_1 \text{ and } \frac{pq}{q+t(p-q)} \in S_2$$

$$\Rightarrow \frac{pq}{q+t(p-q)} \in S_1 \cap S_2, t \in [0,1]$$

$\Rightarrow S_1 \cap S_2$ is a complex harmonic convex set.

Theorem 3.2. *Let S be a complex harmonic convex set and $f : S \to \mathbb{C}$ be a complex harmonic convex function. Then $f = \lambda f$ is also complex harmonic convex function, where* $\lambda \geq 0$.

Proof. Let S be complex harmonic convex set. Then for $p,q \in S, t \in [0,1]$, we have

$$f\left(\frac{pq}{q+t(p-q)}\right) = \lambda f\left(\frac{pq}{q+t(p-q)}\right)$$

$$\leq \lambda\{(1-t)f(p) + tf(q)\}$$

$$= (1-t)\lambda f(p) + t\lambda f(q)$$

$$= (1-t)f(p) + tf(q) \quad [\because \lambda f = f]$$

$\therefore f = \lambda f$ is a complex harmonic convex function.

Theorem 3.3. *Let $f : S \subseteq \mathbb{C}_+ \to \mathbb{C}$ be a complex harmonic convex function on complex harmonic convex set S . Then the level set $S_\lambda = \{ p \in S : f(p) \le \lambda, \lambda \in \mathbb{R} \}$ a complex harmonic convex set.*

Proof. Let $p, q \in S_\lambda$. Then $f(p) \le \lambda, f(q) \le \lambda$.

$$\text{Now } f\left(\frac{pq}{q+t(p-q)}\right) \le (1-t) f(p) + tf(q)$$
$$\le (1-t)\lambda + t\lambda$$
$$= \lambda - t\lambda + t\lambda$$
$$= \lambda$$
$$\therefore f\left(\frac{pq}{q+t(p-q)}\right) \le \lambda \, \forall \, p, q \in S_\lambda$$

$\Rightarrow S_\lambda$ is complex harmonic convex set.

Theorem 3.4. *The function $f : S \subseteq \mathbb{C}_+ \to \mathbb{C}$ is a complex harmonic convex if and only if $E(f)$ is complex harmonic convex set.*

Proof. First, suppose that f is complex harmonic convex function and let $(p, \lambda_1), (p, \lambda_2) \in E(f)$.

Then $f(p) \le \lambda_1, f(q) \le \lambda_2$. For $t \in [0,1]$, we have

$$f\left(\frac{pq}{q+t(p-q)}\right) \le (1-t) f(p) + tf(q)$$
$$\le (1-t)\lambda_1 + t\lambda_2$$
$$\Rightarrow \left(\frac{pq}{q+t(p-q)}, (1-t)\lambda_1 + t\lambda_2\right) \in E(f)$$

$\Rightarrow E(f)$ is complex harmonic convex set

Conversely, suppose $E(f)$ is complex harmonic convex set and let $p, q \in S$.

Then $(p, f(p)), (q, f(q)) \in E(f)$ *, we have*

$$f\left(\frac{pq}{q+t(p-q)}\right) \le (1-t) f(p) + tf(q)$$

$\Rightarrow f$ is complex harmonic convex function.

Theorem 3.5. *[8] Let f and ψ are two complex harmonically convex functions. If f and ψ are similarly ordered, then $f \psi$ is also complex harmonically convex function.*

Proof. Let f and ψ are complex harmonically convex functions. Then

$$f\left(\frac{pq}{q+t(p-q)}\right)\psi\left(\frac{pq}{q+t(p-q)}\right) \le [(1-t) f(p) + tf(q)][(1-t)\psi(p) + t\psi(q)]$$

$$=(1-t)^2 f(p)\psi(p)+t(1-t)f(p)\psi(q)+t(1-t)f(q)\psi(p)+t^2 f(q)\psi(q)$$

$$=(1-t)f(p)\psi(p)+tf(q)\psi(q)+(1-t)^2 f(p)\psi(p)+t^2 f(q)\psi(q)$$

$$+t(1-t)[f(p)\psi(q)+f(q)\psi(p)]-(1-t)f(p)\psi(p)-tf(q)\psi(q)$$

$$\leq(1-t)f(p)\psi(p)+tf(q)\psi(q)$$

$$\therefore f\left(\frac{pq}{q+t(p-q)}\right)\psi\left(\frac{pq}{q+t(p-q)}\right)\leq(1-t)f(p)\psi(p)+tf(q)\psi(q)$$

This proves that product of two complex harmonically convex functions is complex harmonically convex function.

Theorem 3.6. *If $f:S\subseteq\mathbb{C}_+\to\mathbb{C}$ be a complex harmonic quasi convex function and ψ : $\mathbb{C}\to\mathbb{C}$ is increasing function, then $\psi of:S\to\mathbb{C}$ is a complex harmonic quasi convex function.*

Proof. Suppose f is complex harmonic quasi convex function and ψ is increasing function. Then

$$\begin{aligned}(\psi of)\left(\frac{pq}{q+t(p-q)}\right) &=\psi\left[f\left(\frac{pq}{q+t(p-q)}\right)\right]\\ &\leq\psi\left[\max\{f(p),f(q)\}\right]\\ &=\max\{\psi of(p),\psi of(q)\}\\ &=\max\{(\psi of)(p),(\psi of)(q)\}\end{aligned}$$

$\therefore \psi of$ is complex harmonic quasi convex function.

Theorem 3.7. . *If $f:S\subseteq\mathbb{C}_+\to\mathbb{C}$ be a complex harmonic convex function such that $f(p)>f(q)$, then f is complex harmonic pseudo convex function with respect to strictly positive function $\mu(.\,,.)$*

Proof. Suppose $f(p)>f(q)$ and f is complex harmonic convex function. Then

$$\begin{aligned}f\left(\frac{pq}{q+t(p-q)}\right) &\leq(1-t)f(p)+tf(q)\\ &=f(p)-tf(p)+tf(q)\\ &=f(p)+t[f(q)-f(p)]\\ &<f(p)+t(1-t)[f(q)-f(p)]\\ &=f(p)+t(t-1)[f(p)-f(q)]\\ &<f(p)+t(t-1)\mu(q,p)\quad \because \mu(q,p)=f(p)-f(q)>0\end{aligned}$$

Theorem 3.8. *Let $f : S \subseteq \mathbb{C}_+ \to \mathbb{C}$ be a differentiable on a non empty complex harmonic convex set S. Then f is complex harmonic quasi convex*

$$\Leftrightarrow f(p) \le f(q) \Rightarrow \left\langle f'(q), \frac{pq}{q-p} \right\rangle \le 0, \ \forall p, q \in S$$

Proof. First, suppose f is complex harmonic quasi-convex and $p, q \in S$ such that $f(p) \le f(q)$

Then by Taylorseries,

$$f\left(\frac{pq}{q+t(p-q)}\right) = f(q) + t\left\langle f'(q), \frac{pq}{q-p} \right\rangle + t \| \frac{pq}{q-p} \| \alpha\left[q; t\left(\frac{pq}{q-p}\right)\right] \qquad (3.1)$$

As $t \to 0$, $\alpha\left[q; t\left(\frac{pq}{q-p}\right)\right] \to 0$

Since the function is complex harmonic quasi convex, therefore,

$$f\left(\frac{pq}{q+t(p-q)}\right) \le f(q)$$

Hence, from equation (3.1), we get

$$\Rightarrow t\left\langle f'(q), \frac{pq}{q-p} \right\rangle + t \| \frac{pq}{q-p} \| \alpha\left[q; t\left(\frac{pq}{q-p}\right)\right] \le 0$$

$$\Rightarrow \left\langle f'(q), \frac{pq}{q-p} \right\rangle + t \| \frac{pq}{q-p} \| \alpha\left[q; t\left(\frac{pq}{q-p}\right)\right] \le 0$$

As $t \to 0$, we have, $\left\langle f'(q), \frac{pq}{q-p} \right\rangle \le 0$.

Conversely, suppose that p, q $\in S$ and $f(p) \le f(q)$.

We need to show that $f\left(\frac{pq}{q+t(p-q)}\right) \le f(q), \forall\, p, q \in S, t \in (0,1)$

For this, we have to show that the set

$$M = \left\{ p' : p' = \frac{pq}{q+t(p-q)}, t \in (0,1),\ f(p') > f(q) \right\} \text{ is empty}$$

On contrary, suppose that there exist

$$p' \in M : p' = \frac{pq}{q+t(p-q)}, t \in (0,1),\ f(p') > f(q)$$

Since f is differentiable $\Rightarrow f$ is continuous and hence there exists $\delta \in (0,1)$ such that $f\left(\frac{p'q}{q+\eta(p-q)}\right) > f(q)$ for each $\eta \in (\delta, 1)$ and $f(p') > f\left(\frac{p'q}{q+\delta(p'-q)}\right)$.

By Mean Value Theorem, we have

$$0 < f(p') - f\left(\frac{p'q}{q+\delta(p'-q)}\right) = (1-t)\left\langle f'(\tilde{p}), \frac{p'q}{q-p'} \right\rangle$$

(3.2)

where $\tilde{p}=\frac{p'q}{q+\eta'(p'-q)}$ for some $\eta'\in(\delta,1)$,

$$\Rightarrow f(\tilde{p})>f(q) \tag{3.3}$$

From equation (3.2), we have

$$\left\langle f'(\tilde{p}),\frac{p'q}{q-p'}\right\rangle=\frac{f(p')-f(\frac{p'q}{q+\delta(p'-q)}}{1-t}>0$$

$$\Rightarrow\left\langle f'(\tilde{p}),\frac{p'q}{q-p'}\right\rangle>0$$

From equation (3.3), we have

$f(\tilde{p})>f(q)\geq f(p)$ and $\tilde{p}$ is a harmonic combination of p and q

By given condition $\left\langle f'(\tilde{p}),\frac{\tilde{p}p}{\tilde{p}-p}\right\rangle\leq 0$ and thus we must have $\left\langle f'(\tilde{p}),\frac{pp}{q-p}\right\rangle\leq 0$

This inequality is not compatible with (3.4).

Therefore, $M=\phi$. Hence the proof.

IV. Conclusion

In this paper, we have introduced and studied a new class of harmonic convex and harmonic quasi convex functions in complex plane. We have also derived some basic results and properties involving harmonic convex and harmonic quasi convex functions in complex plane.

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